A LOCALLY MINIMAL, BUT NOT GLOBALLY MINIMAL BRIDGE POSITION OF A KNOT

MAKOTO OZAWA AND KAZUTO TAKAO

Abstract. We present a locally minimal, but not globally minimal bridge position of a knot, that is, an unstabilized, nonminimal bridge position of a knot. It implies that a bridge position cannot always be simplified so that the bridge number monotonically decreases to the minimal.

1. Introduction

A *knot* is an equivalent class of embeddings of the circle S^1 into the 3-sphere S^3, where two embeddings are said to be *equivalent* if an ambient isotopy of S^3 deforms one to the other. In knot theory, it is a fundamental and important problem to determine whether two given representatives of knots are equivalent, and furthermore to describe how one can be deformed to the other. In particular, a simplification to a "minimal position" is of great interest.

Let $h : S^3 \to \mathbb{R}$ be the standard Morse function, that is, the restriction of $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ to S^3. A *Morse position* of a knot K is a representative k such that k is disjoint from the poles of S^3 and the critical points of $h|_k$ are all non-degenerate and have pairwise distinct values. Since k is a circle, there are the same number of local maxima and local minima.

In [19], Schubert introduced the notion of bridge position and bridge number for knots. A *bridge position* of K is a Morse position k where all the local maxima are above all the local minima with respect to h. A level 2-sphere S separating the local maxima from the local minima is called a *bridge sphere* of k. If k intersects S in $2n$ points, then k is called an n*-bridge position* and n is called the *bridge number* of k. The minimum of the bridge number over all bridge positions of K is called the *bridge number* of K. A knot with the bridge number n is called an n*-bridge knot*. The bridge number is a fundamental geometric invariant of knots as well as the crossing number.

In [6], Gabai introduced the notion of width for knots. Suppose that k is a Morse position of a knot K, let $t_1, \dots, t_m$ be the critical levels of $h|_k$ such that $t_i < t_{i+1}$ for $i = 1, \dots, m-1$, and choose regular levels $r_1, \dots, r_{m-1}$ of $h|_k$ so that $t_i < r_i < t_{i+1}$. The *width* of k is defined as $\sum_{i=1}^{m-1} |h^{-1}(r_i) \cap k|$, and the *width* of K is the minimum of the width over all Morse positions of K.

Two Morse positions of a knot are *isotopic* if an ambient isotopy of S^3 deforms one to the other keeping it a Morse position except for exchanging two levels of local maxima or two levels of local minima. Such an isotopy preserves the width

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of a Morse position and the bridge number of a bridge position. The following two types of moves change the isotopy class of a Morse position.

Suppose that k is a Morse position of a knot K and let $t_1, r_1, t_2, \ldots, r_{m-1}, t_m$ as above. We say that the level 2-sphere $h^{-1}(r_i)$ is *thick* if t_i is a locally minimal level of $h|_k$ and t_{i+1} is a locally maximal level of $h|_k$, and that $h^{-1}(r_i)$ is *thin* if t_i is a locally maximal level and t_{i+1} is a locally minimal level. A *strict upper* (resp. *lower*) *disk* for a thick sphere S is a disk $D \subset S^3$ such that the interior of D does not intersect with k and any thin sphere, the interior of D contains no critical points with respect to h, and ∂D consists of a subarc α of k and an arc in $S - k$. Note that the arc α has exactly one local maximum (resp. minimum). First suppose that there exist a strict upper disk D_+ and a strict lower disk D_- for a thick sphere S such that $D_+ \cap D_-$ consists of a single point of $k \cap S$. Then k can be isotoped along D_+ and D_- to cancel the local maximum in ∂D_+ and the local minimum in ∂D_-. In [21], Schultens called this move a *Type I move*. The inverse operation of a Type I move is called a *stabilization* ([2], [12]) or a *perturbation* ([18], [23]) and the resultant position is said to be *stabilized* or *perturbed*. Next suppose that there exist a strict upper disk D_+ and a strict lower disk D_- for a thick sphere S such that $D_+ \cap D_- = \emptyset$. Then k can be isotoped along D_+ and D_- to exchange the two levels of the local maximum in ∂D_+ and the local minimum in ∂D_-. Schultens ([21]) called this move a *Type II move*.

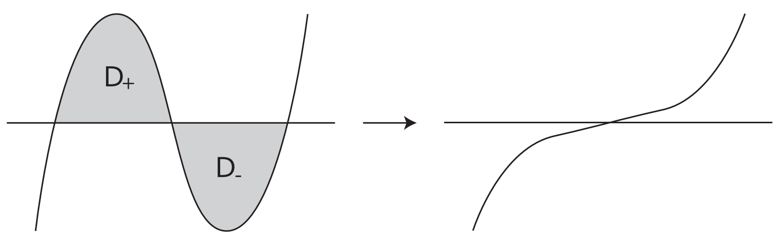

FIGURE 1. Type I move

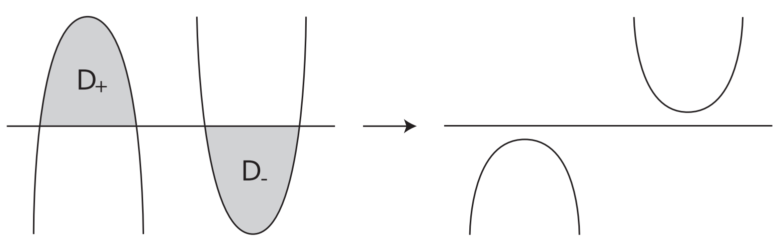

FIGURE 2. Type II move

The followings are fundamental theorems corresponding to Reidemeister's theorem ([16], [1]) that two knots are equivalent if and only if their two knot diagrams can be related by a sequence of three types of moves, so-called Reidemeister moves.

Theorem 1.1 ([2], [7])**.** *Two knots are equivalent if and only if their two bridge positions can be related by a sequence of Type I moves and the inverse operations up to isotopy.*

Theorem 1.2 ([21])**.** *Two knots are equivalent if and only if their two Morse positions can be related by a sequence of Type I and Type II moves and the inverse operations up to isotopy.*

We say that a bridge position of a knot K is *globally minimal* if it realizes the bridge number of K, and a bridge position is *locally minimal* if it does not admit a Type I move. Similarly, we say that a Morse position of a knot K is *globally minimal* if it realizes the width of K, and a Morse position is *locally minimal* if it does not admit a Type I move nor a Type II move. Note that if a bridge (resp. Morse) position of a knot is globally minimal, then it is locally minimal. Otal (later Hayashi–Shimokawa, the first author) showed the converse for bridge positions of the trivial knot.

Theorem 1.3 ([12], [8], [14])**.** *A locally minimal bridge position of the trivial knot is globally minimal.*

It implies that whatever complicated bridge position of the trivial knot can be simplified into the 1-bridge position only by Type I moves. Furthermore, Otal (later Scharlemann–Tomova) showed that the same statement for 2-bridge knots is true ([13], [18]), and the first author showed that the same statement for torus knots is also true ([15]). Then, the following problem is naturally proposed.

Problem 1.4 ([15])**.** *Is any locally minimal bridge position of any knot is globally minimal?*

In this paper, we give a negative answer to this problem. It implies that a bridge position cannot always be simplified into a minimal bridge position only by Type I moves.

Theorem 1.5. *A* 4*-bridge position* κ *of a knot* $\mathcal{K}$ *in Figure 3 is locally minimal, but not globally minimal.*

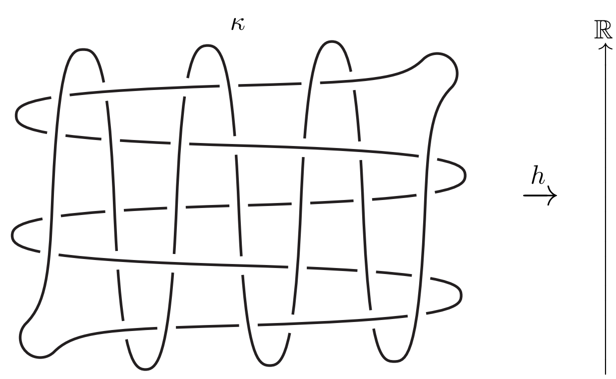

FIGURE 3. A 4-bridge position of a knot.

To prove this theorem, we show that the Hempel distance of the 4-bridge position is greater than 1 by the method developed by the second author ([22]). This guarantees that the 4-bridge position is locally minimal (see Lemma 2.1).

On the other hand, Zupan showed that locally minimal, but not globally minimal Morse positions exist even if the knot is trivial.

Theorem 1.6 ([26]). *There exists a locally minimal Morse position of the trivial knot which is not globally minimal.*

We remark that this answers Scharlemann's question [17, Question 3.5]. By using this example, Zupan showed that there exist infinitely many locally minimal, but not globally minimal Morse positions for any knot.

2. Proof of the main theorem

Note that Figure 3 displays a 3-bridge position of $\mathcal{K}$ after a $(\pi/2)$-rotation of κ, and so the 4-bridge position is not globally minimal. To prove that the 4-bridge position is locally minimal, we apply the followings:

Lemma 2.1. *An n-bridge position is locally minimal if it has the Hempel distance greater than* 1.

Theorem 2.2 ([22]). *An n-bridge position has the Hempel distance greater than* 1 *if a bridge diagram of it satisfies the well-mixed condition.*

Here n is an integer greater than 2. The notions *Hempel distance*, *bridge diagram* and *well-mixed condition* are described in the following subsections.

2.1. **Hempel distance.** Suppose that k is an n-bridge position of a knot K and S is a bridge sphere of k. Let $B_+,\ B_- \subset S^3$ be the 3-balls divided by S, and τ_ε be the n arcs $k \cap B_\varepsilon$ for each $\varepsilon = \pm$.

Consider a properly embedded disk E in B_ε. We call E an *essential disk* of $(B_\varepsilon, \tau_\varepsilon)$ if E is disjoint from τ_ε and ∂E is essential in the $2n$-punctured sphere $S \setminus k$. Here, a simple closed curve on a surface is said to be *essential* if it neither bounds a disk nor is peripheral in the surface. The essential simple closed curves on $S \setminus k$ form a 1-complex $\mathcal{C}(S \setminus k)$, called the *curve graph* of $S \setminus k$. The vertices of $\mathcal{C}(S \setminus k)$ are the isotopy classes of essential simple closed curves on $S \setminus k$, and a pair of vertices spans an edge of $\mathcal{C}(S \setminus k)$ if the corresponding isotopy classes can be realized as disjoint curves. The *Hempel distance* of k is defined as

$$\min\{d([\partial E_+], [\partial E_-]) \mid E_\varepsilon \text{ is an essential disk of } (B_\varepsilon, \tau_\varepsilon) \text{ for each } \varepsilon = \pm.\},$$

where $d([\partial E_+], [\partial E_-])$ is the minimal distance between $[\partial E_+]$ and $[\partial E_-]$ measured in $\mathcal{C}(S \setminus k)$ with the path metric.

Assume that k has the Hempel distance 0. By the definition, there exist essential disks $E_+,\ E_-$ of $(B_+, \tau_+),\ (B_-, \tau_-)$, respectively, such that $\partial E_+ = \partial E_-$, which requires that k is split. Since the circle k is connected, the Hempel distance is at least 1. The Hempel distance is 1 if there exist essential disks $E_+,\ E_-$ of $(B_+, \tau_+),\ (B_-, \tau_-)$, respectively, such that $\partial E_+ \cap \partial E_- = \emptyset$. We can find such disks for a not locally minimal bridge position as follows:

Proof of Lemma 2.1. Assume that the n-bridge position k is not locally minimal. By definition, there exist a strict upper disk $D_+ \subset B_+$ and a strict lower disk $D_-^1 \subset B_-$ for S such that $D_+ \cap D_-^1$ consists of a single point of $k \cap S$. Note that τ_- is n arcs each of which has a single local minimum. We can choose strict lower disks $D_-^2,\ \ldots,\ D_-^n \subset B_-$ for S such that $D_-^1,\ D_-^2,\ \ldots,\ D_-^n$ are pairwise disjoint. Let $\eta(D_+ \cup D_-^1)$ denote a closed regular neighborhood of $D_+ \cup D_-^1$ in S^3. By replacing subdisks of $D_-^2,\ \ldots,\ D_-^n$ with subdisks of $\partial(\eta(D_+ \cup D_-^1)) \cap B_-$, we can arrange that $D_-^1,\ D_-^2,\ \ldots,\ D_-^n$ are disjoint from D_+ except for the two points of $k \cap S$. Since we assumed $n > 2$, one of the strict lower disks, denoted by D_-, is disjoint

from D_+. The boundary of a regular neighborhood in S^3 of each D_ε intersects B_ε in an essential disk of $(B_\varepsilon, \tau_\varepsilon)$. They guarantee that the Hempel distance is 1. □

2.2. **Bridge diagram.** We continue with the above notations. There are pairwise disjoint n strict upper (resp. lower) disks $D_+^1,\ D_+^2,\ \ldots,\ D_+^n$ (resp. $D_-^1,\ D_-^2,\ \ldots,$ D_-^n) for S. The knot diagram of K obtained by projecting k into S along these disks is called a *bridge diagram* of k. In the terminology of [5], $\tau_+,\ \tau_-$ are the overpasses and the underpasses of k.

Now let us describe how we can obtain a bridge diagram of the 4-bridge position κ. Isotope κ as in Figure 4, and start with a bridge sphere $S = S_0$. There are canonical strict upper disks $D_+^1,\ D_+^2,\ D_+^3$ and D_+^4. Figure 5 illustrates a view of the arcs $D_+^1 \cap S,\ D_+^2 \cap S,\ D_+^3 \cap S$ and $D_+^4 \cap S$ on S from B_+ side. Shifting the bridge sphere S to S_1, the arcs can be seen as in Figure 6. Shifting S further to S_2 and to S_3, the arcs are as in Figure 7 and 8, respectively. By continuing this process, the arcs are as in Figure 9 when S is at S_{15}. The picture goes complicated and complicated as S goes down. We include huge pictures in the back of this paper. Figure 10 illustrates the arcs when S is at S_{20}, and finally Figure 11 when S has arrived at S_{25}. Then we can find canonical strict lower disks $D_-^1,\ D_-^2,\ D_-^3,\ D_-^4$ and obtain a bridge diagram of κ.

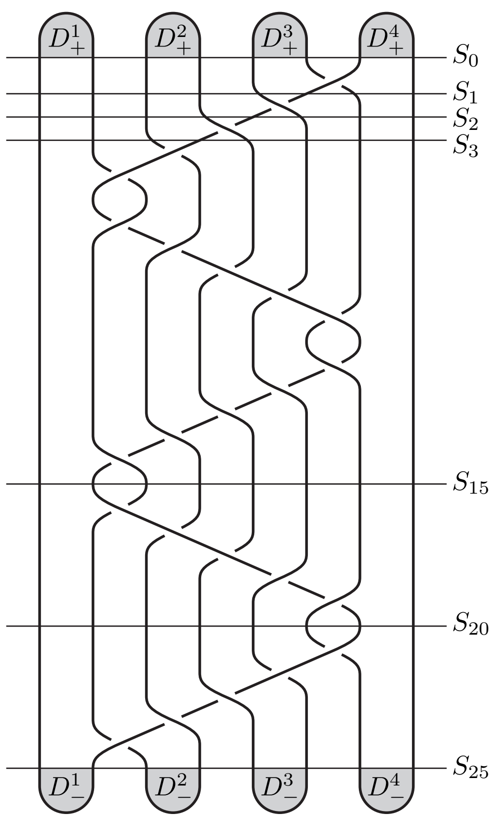

FIGURE 4. A 4-bridge position isotopic to κ.

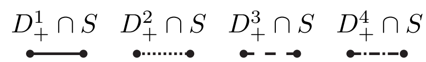

FIGURE 5. The arcs in S_0.

FIGURE 6. In S_1.

FIGURE 7. In S_2.

FIGURE 8. In S_3.

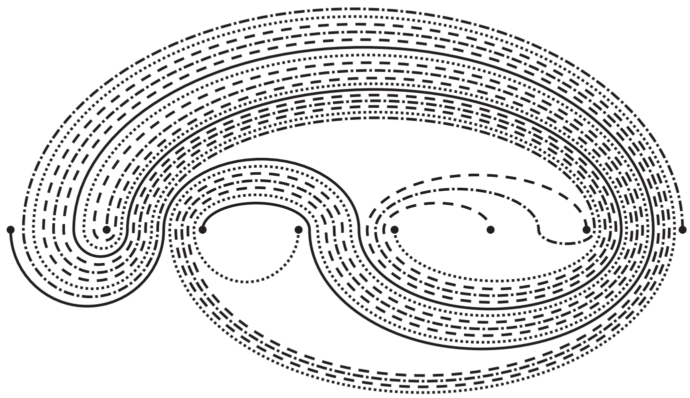

FIGURE 9. In S_{15}.

2.3. **Well-mixed condition.** Suppose again that k is an n-bridge position of a knot K with $n > 2$ and S is a bridge sphere of k. Let $B_+,\ B_- \subset S^3$ be the 3-balls divided by S, and τ_ε be the n arcs $K \cap B_\varepsilon$ for each $\varepsilon = \pm$. Let $D_+^1,\ D_+^2,\ \ldots,\ D_+^n$ and $D_-^1,\ D_-^2,\ \ldots,\ D_-^n$ be strict upper and lower disks for S determining a bridge diagram of k.

Let l be a loop on S containing the arcs $D_-^1 \cap S,\ D_-^2 \cap S,\ \ldots,\ D_-^n \cap S$ such that the arcs are located in l in that order. We can assume that $D_+^1,\ D_+^2,\ \ldots,\ D_+^n$ have been isotoped so that the arcs $D_+^1 \cap S,\ D_+^2 \cap S,\ \ldots,\ D_+^n \cap S$ have minimal intersection with l. For the bridge diagram of Figure 11, it is natural to choose l to be the one which is seen as a horizontal line. Let $H_+,\ H_- \subset S$ be the hemi-spheres divided by l and let δ_i $(1 \le i \le n)$ be the component of $l \setminus (D_-^1 \cup D_-^2 \cup \cdots \cup D_-^n)$ which lies between $D_-^i \cap S$ and $D_-^{i+1} \cap S$. (Here the indices are considered modulo n.) Let $\mathcal{A}_{i,j,\varepsilon}$ be the collection of components of $(D_+^1 \cup D_+^2 \cup \cdots \cup D_+^n) \cap H_\varepsilon$ separating δ_i from δ_j in H_ε for a distinct pair $i,\ j \in \{1, 2, \ldots, n\}$ and $\varepsilon \in \{+, -\}$. For example, Figure 12 roughly displays $\mathcal{A}_{1,2,+}$ for the bridge diagram of Figure 11. Note that $\mathcal{A}_{i,j,\varepsilon}$ consists of parallel arcs in H_ε.

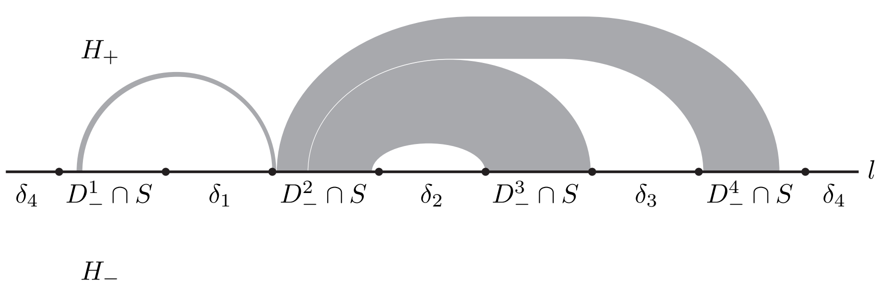

FIGURE 12. The collection $\mathcal{A}_{1,2,+}$ of arcs, which looks gray bands.

Definition 2.3. (1) A bridge diagram satisfies the (i, j, ε)-*well-mixed condition* if in $\mathcal{A}_{i,j,\varepsilon} \subset H_\varepsilon$, a subarc of $D_+^r \cap S$ is adjacent to a subarc of $D_+^s \cap S$ for all distinct pair $r,\ s \in \{1, 2, \ldots, n\}$.

(2) A bridge diagram satisfies the *well-mixed condition* if it satisfies the (i, j, ε)-well-mixed condition for all combinations of a distinct pair $i,\ j \in \{1, 2, \dots, n\}$ and $\varepsilon \in \{+, -\}$.

By making Figure 12 detailed, one can check the $(1, 2, +)$-well-mixed condition for the bridge diagram of Figure 11. One can also check the (i, j, ε)-well-mixed condition for all the other (i, j, ε) to complete the well-mixed condition. By Theorem 2.2, the Hempel distance of κ is greater than 1. By Lemma 2.1, we conclude the proof of Theorem 1.5.

We would like to remark that the Hempel distance of κ is exactly 2. Notice that the boundary of a regular neighborhood in S of the closure of δ_3 is a simple closed curve disjoint from both D^1_+ and D^1_-. Note that the boundary of a regular neighborhood in S^3 of each D^1_ε intersects B_ε in an essential disk of $(B_\varepsilon, \tau_\varepsilon)$. They guarantee that the Hempel distance is at most 2.

3. Related results and further directions

3.1. **The knot of our example.** Figure 3 shows that the bridge number of $\mathcal{K}$ is at most 3. Since any locally minimal bridge position of any 2-bridge knot is globally minimal ([13], [18]), the bridge number of $\mathcal{K}$ is equal to 3. Since $\mathcal{K}$ has a 4-bridge position with the Hempel distance 2, it is a prime knot by the following:

Theorem 3.1. *A bridge position of a composite knot has the Hempel distance* 1.

Proof. Let k be an n-bridge position of a composite knot and S be a bridge sphere of k. Let $B_+,\ B_- \subset S^3$ be the 3-balls divided by S, and τ_ε be the n arcs $k \cap B_\varepsilon$ for each $\varepsilon = \pm$. By the arguments in [19], [20], it follows that any decomposing sphere for a bridge position of a composite knot can be isotoped so that it intersects a bridge sphere in a single loop. Then, in the opposite sides of the decomposing sphere, there are two essential disks $E_+,\ E_-$ of $(B_+, \tau_+),\ (B_-, \tau_-)$ respectively such that $\partial E_+ \cap \partial E_- = \emptyset$. This shows that the Hempel distance is 1. $\square$

Furthermore, $\mathcal{K}$ is hyperbolic since any locally minimal bridge position of any torus knot is globally minimal ([15]). Thus, $\mathcal{K}$ is a hyperbolic 3-bridge knot which admits a 4-bridge position with the Hempel distance 2.

We expect that not only κ may be an example for Theorem 1.5 but also many 4-bridge positions of knots with the same projection image as that of Figure 3. However only finitely many knots have the same projection image, and we would like to ask the following problem.

Problem 3.2. *For an integer $n > 3$, can we generate infinitely many n-bridge positions which is locally minimal, but not globally minimal?*

We further expect that for some integers $n > m \geq 3$, we can find a locally minimal n-bridge position of an m-bridge knot which has a similar projection image as that of Figure 3. However it seems difficult to find more than two locally minimal bridge positions of such a knot, and we would like to ask the following problem.

Problem 3.3. *Does any knot have infinitely many locally minimal bridge positions?*

It should be remarked that multiple bridge surfaces restrict Hempel distances ([24]), and there exist only finitely many bridge positions of given bridge numbers for a hyperbolic knot ([4]). By Coward's theorem ([4]), there are finitely many globally minimal bridge positions of a knot.

3.2. **Essential surfaces.** Composite knots are the most simple example of knots with essential surfaces properly embedded in the exteriors of their representatives. Theorem 3.1 suggests that essential surfaces restrict Hempel distances. Bachman–Schleimer showed it in general.

Theorem 3.4 ([3])**.** *Let F be an orientable essential surface properly embedded in the exterior of a bridge position k of a knot. Then the Hempel distance of k is bounded above by twice the genus of F plus $|\partial F|$.*

By Theorem 3.4, if a knot exterior contains an essential annulus or an essential torus, then the Hempel distance of a bridge position is at most 2. Therefore, if there exists a bridge position of a knot with the Hempel distance at least 3, then the knot is hyperbolic. The properties of our knot $\mathcal{K}$ can be compared with it.

A knot without an essential surface with meridional boundary in the exterior of its representative is called a *meridionally small knot*. For example, the trivial knot, 2-bridge knots and torus knots are known to be meridionally small. As we mentioned in Section 1, these knots also have the nice property that any nonminimal bridge position is stabilized. We say that a knot is *destabilizable* if it has this property. Zupan showed that any cabled knot J of a meridionally small knot K is also meridionally small, and that if K is destabilizable, then J is also destabilizable ([25]). Then, the following problem is naturally proposed.

Problem 3.5. *Are there some relation between meridionally small knots and destabilizable knots?*

We remark that a bridge position of a meridionally small knot is locally minimal if and only if the Hempel distance is greater than 1 by Lemma 2.1 and the following fundamental result:

Theorem 3.6 ([9], [23])**.** *If a bridge position of a knot has the Hempel distance* 1*, then either it is stabilized or the knot exterior contains an essential surface with meridional boundary.*

On the other hand, it is not always true that if the knot exterior contains an essential surface with meridional boundary, then a bridge position has the Hempel distance 1. For example, [10, Example 5.1] shows that a 3-bridge position of 8_{16} has the Hempel distance greater than 1, but the knot exterior contains an essential surface with meridional boundary.

3.3. **Distance between bridge positions.** Theorem 1.1 allows us to define a distance between two bridge positions of a knot, which we call the *Birman distance*. That is to say, the Birman distance between two bridge positions is the minimum number of Type I moves and the inverse operations relating the bridge positions up to isotopies. For example, the Birman distance between an n-bridge position and an m-bridge position of the trivial knot is always $|n-m|$ by Theorem 1.3. The Birman distance between κ and the 3-bridge position of $\mathcal{K}$ is at least 3 since κ is locally minimal. In fact, we can see that it is at most 5 by observing the $(\pi/2)$-rotation of κ.

Johnson–Tomova gave an upper bound for the Birman distance between two bridge positions with high Hempel distance which are obtained from each other by flipping, namely the rotation of S^3 exchanging the poles.

Theorem 3.7 ([11])**.** *For an integer $n \geq 3$, if an n-bridge position k of a prime knot has the Hempel distance at least $4n$, then the Birman distance between k and the flipped bridge position of k is $2n-2$.*

They also gave the following, which holds even if we consider bridge positions modulo flipping.

Theorem 3.8 ([11])**.** *For an integer $n \geq 3$, there exists a composite knot with a $2n$-bridge position and a $(2n-1)$-bridge position such that the Birman distance is at least $2n-7$.*

We remark that the $2n$-bridge position is not locally minimal, and hence it does not answer Problem 1.4. It turns out that there are two $(2n-1)$-bridge positions such that the Birman distance is greater than or equal to $2n-6$. The followings are major problems.

Problem 3.9. *Determine or estimate the Birman distance in terms of some invariants of the bridge positions.*

Problem 3.10. *For a given n, does there exist a universal upper bound for the Birman distance between locally minimal bridge positions of every n-bridge knot?*

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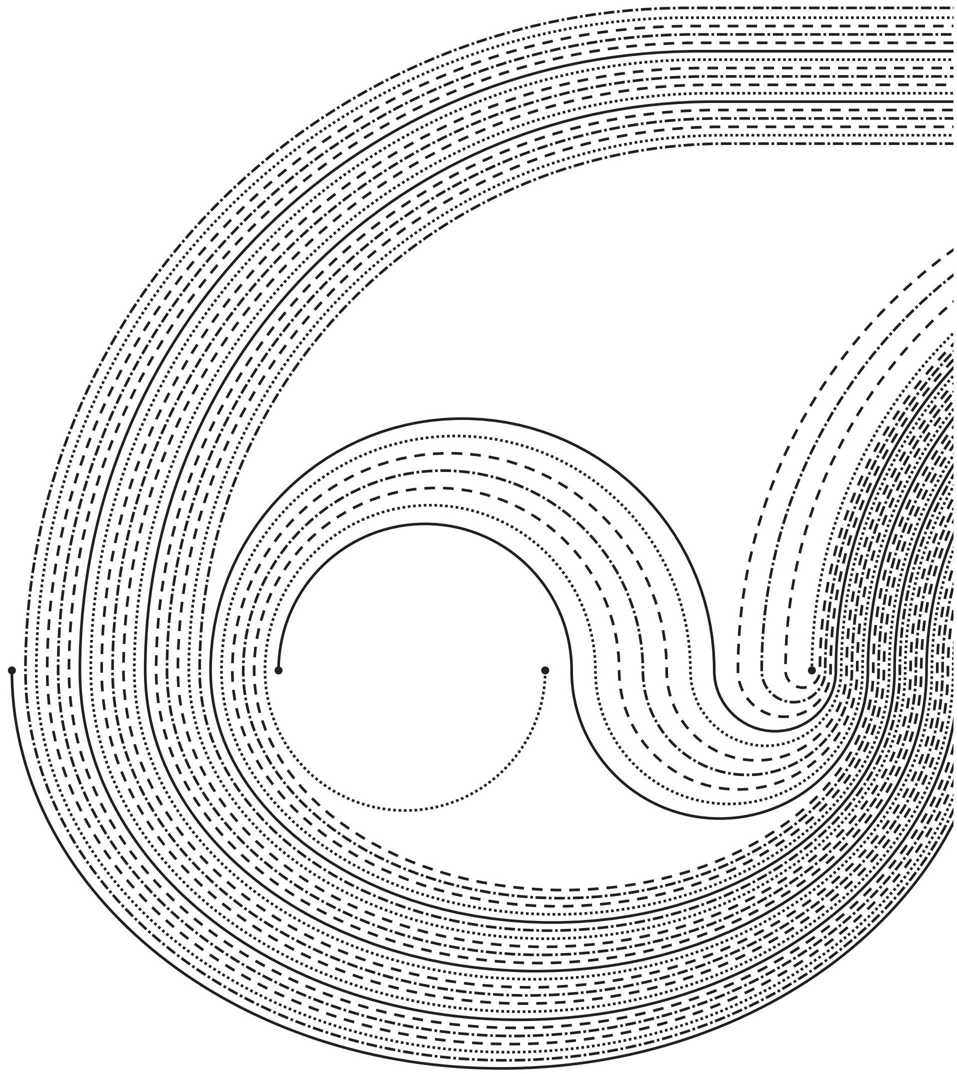
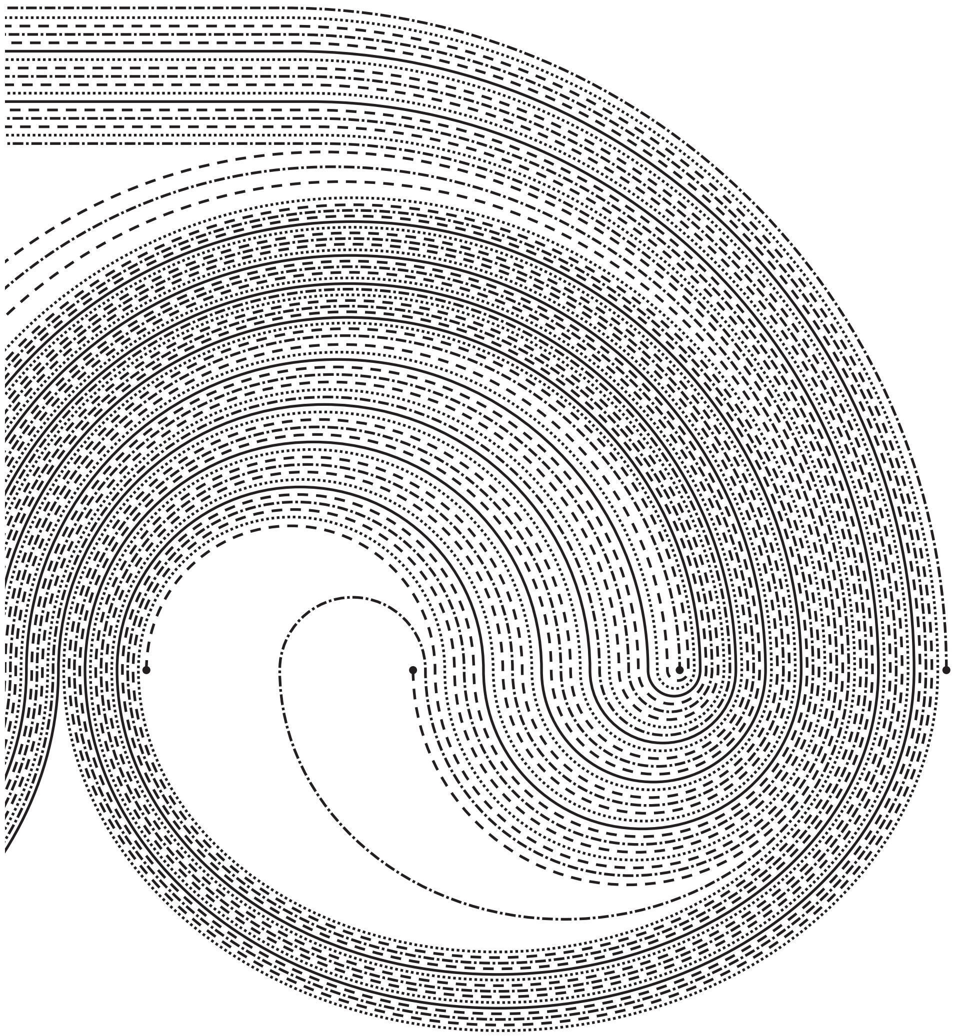


FIGURE 10. The arcs $D^1_+ \cap S$, $D^2_+ \cap S$, $D^3_+ \cap S$ and $D^4_+ \cap S$ in $S = S_{20}$, which extend to the next page.



The right part of Figure 10.

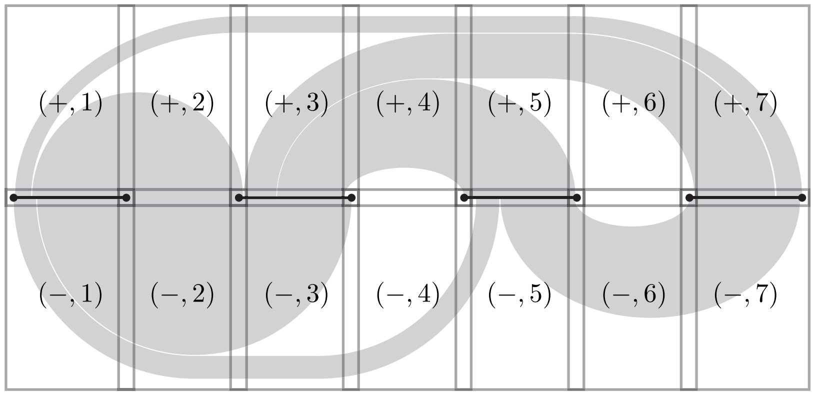
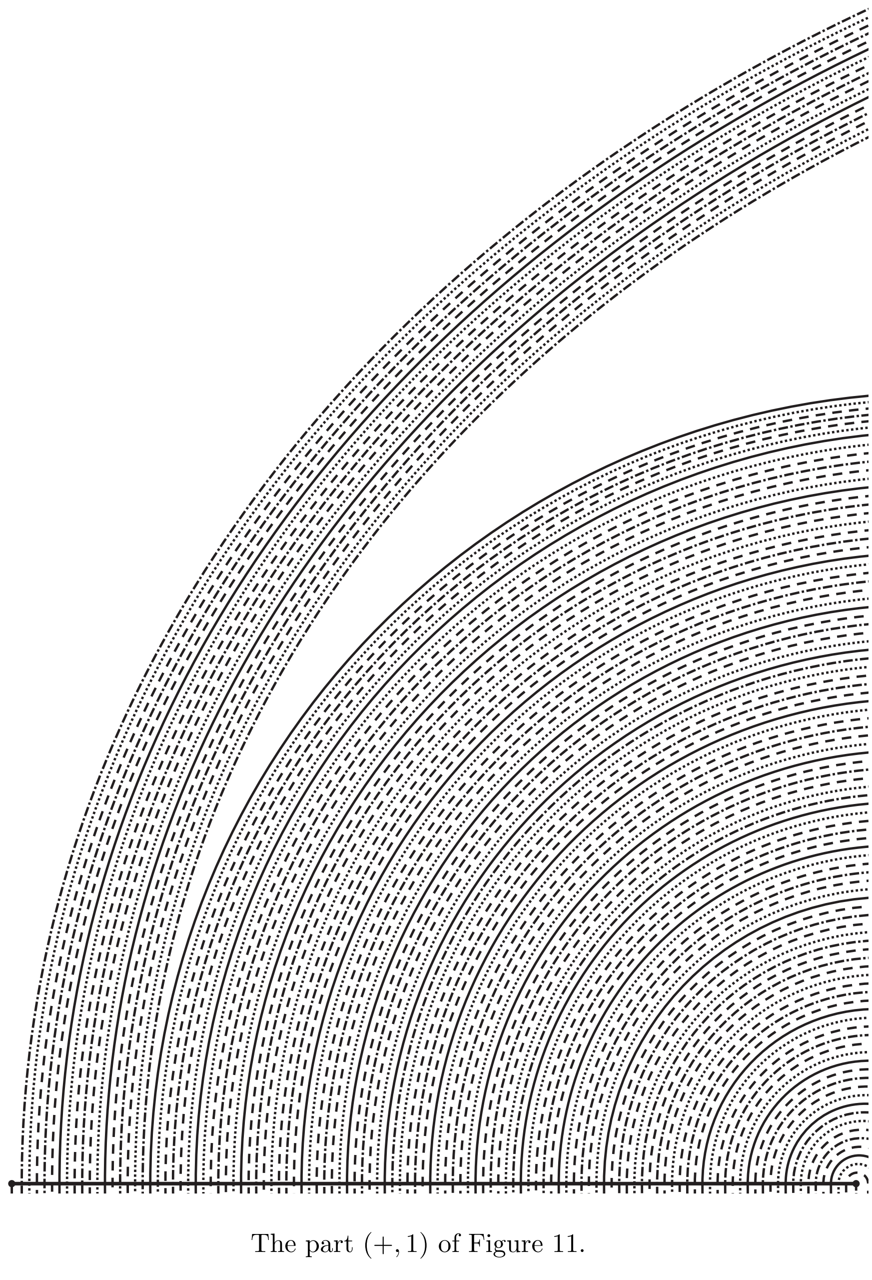
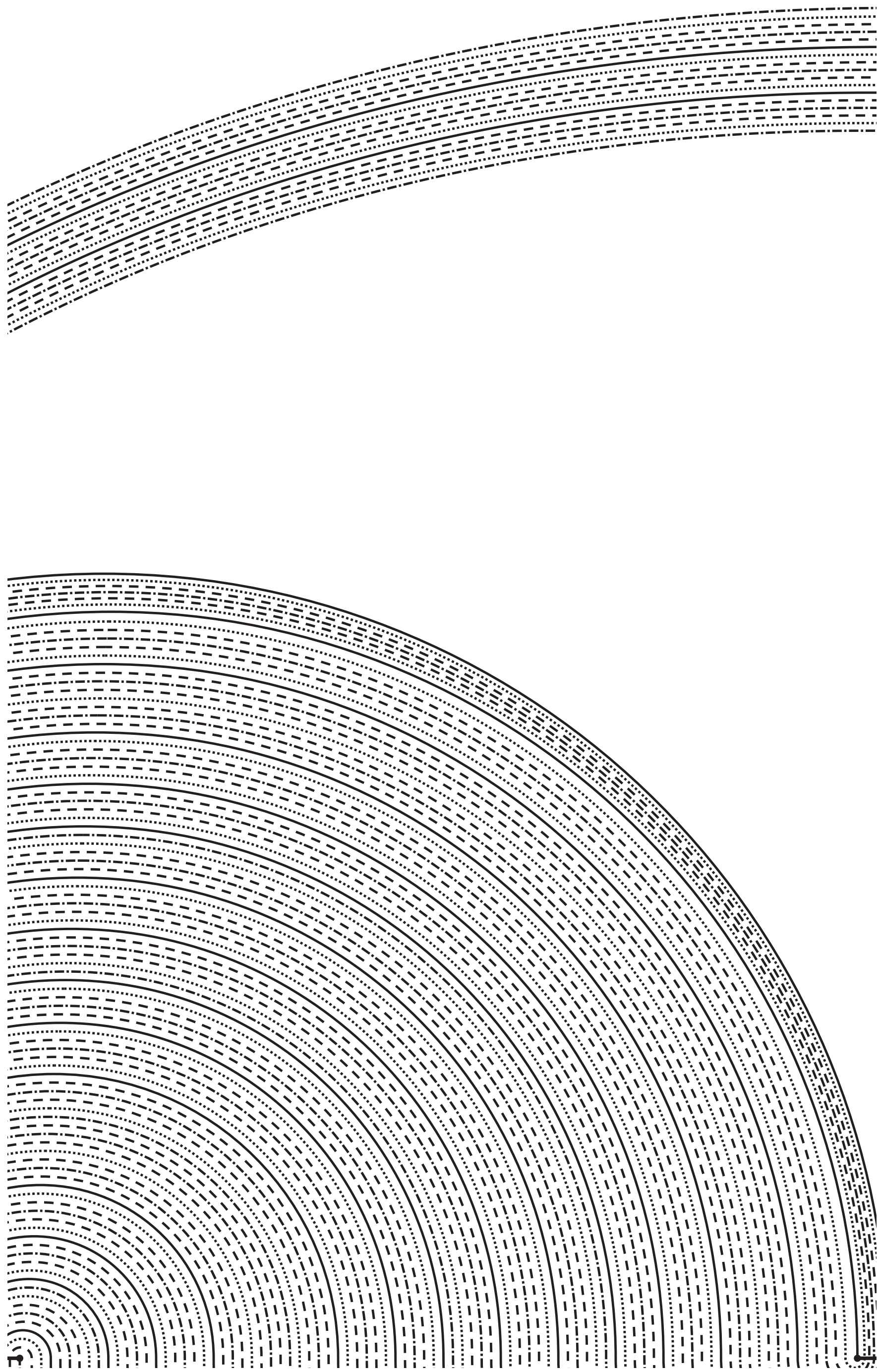


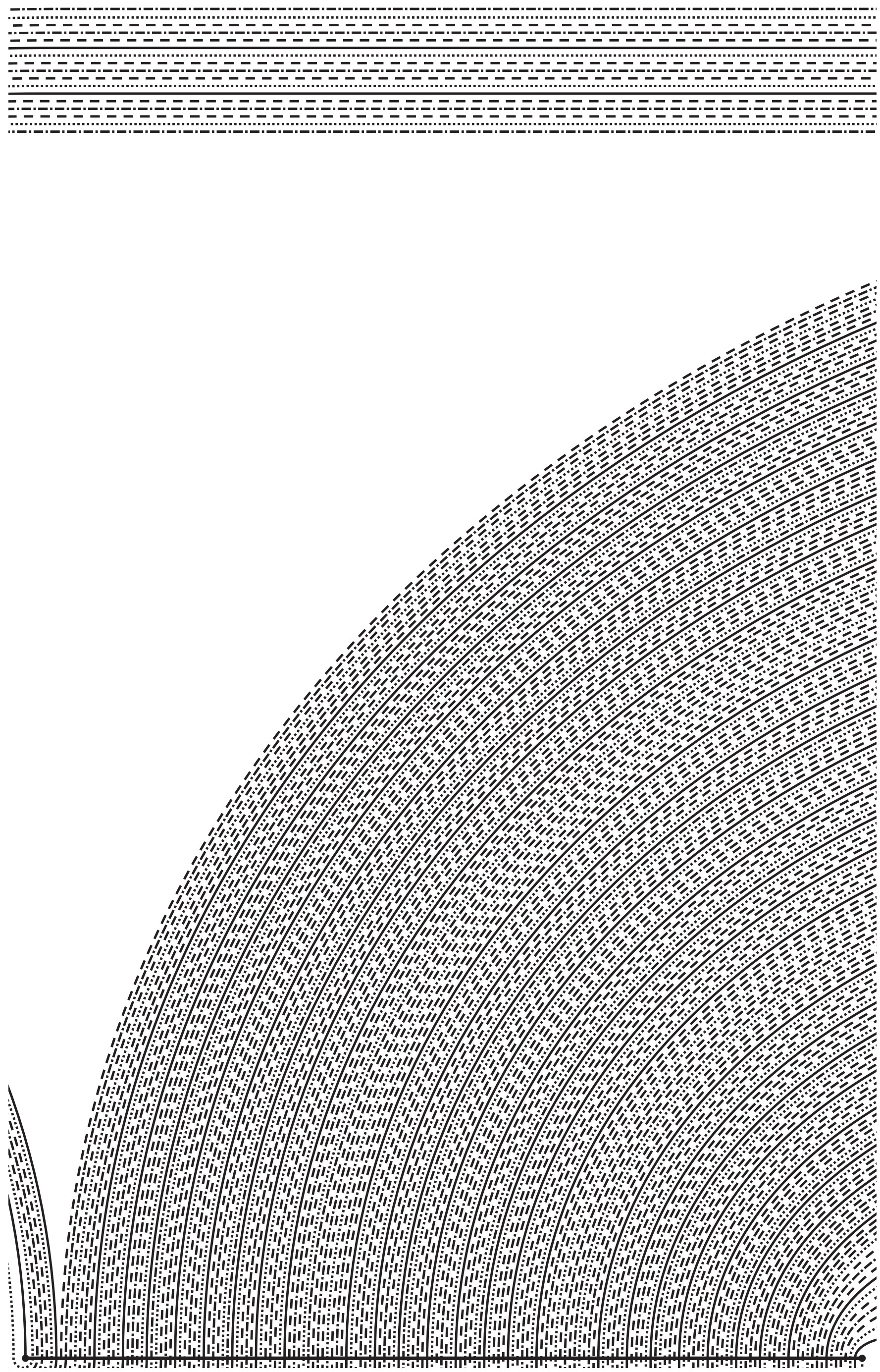
FIGURE 11. A bridge diagram of the 4-bridge position, which is decomposed into the following 14 pages.



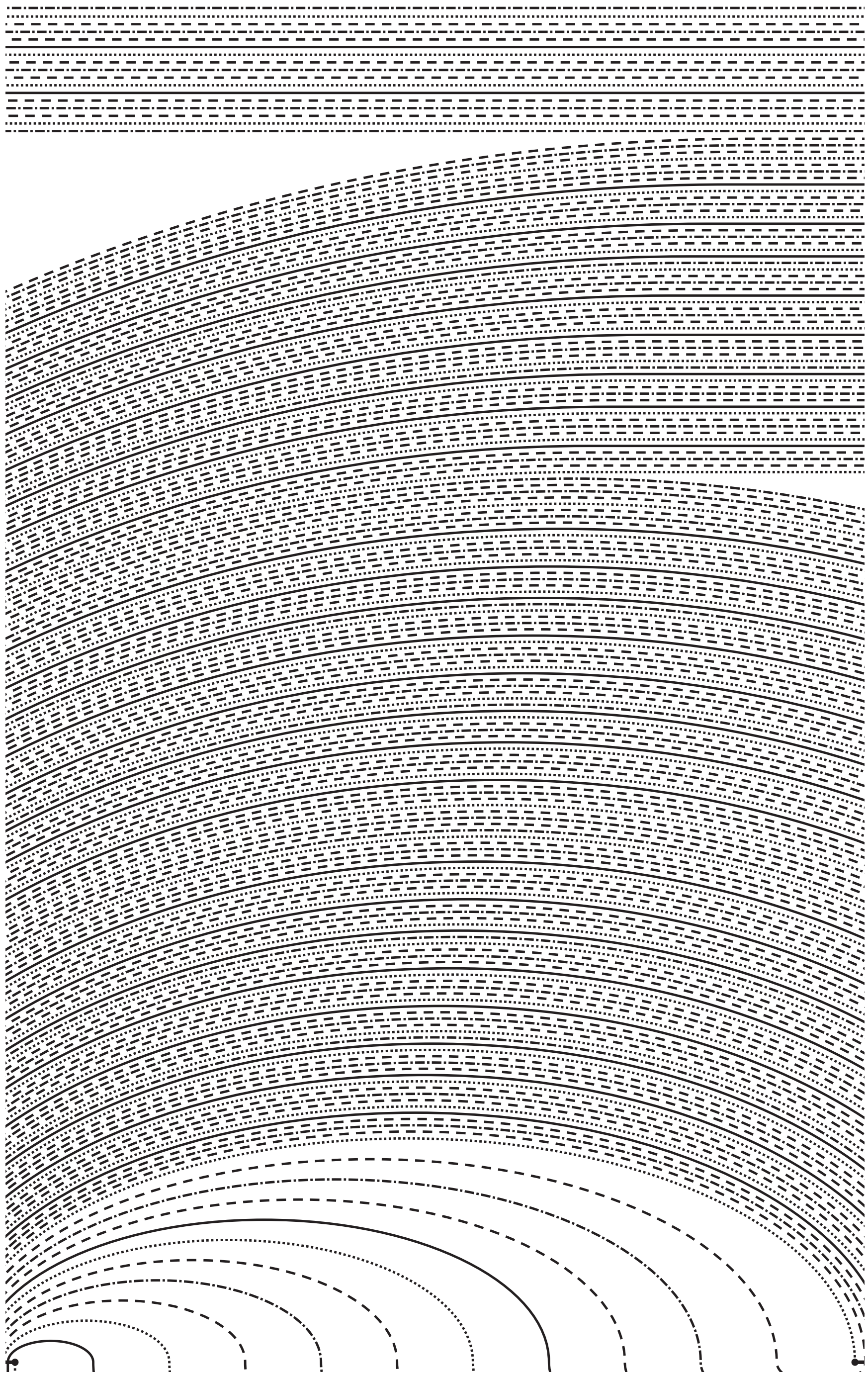
The part $(+, 1)$ of Figure 11.



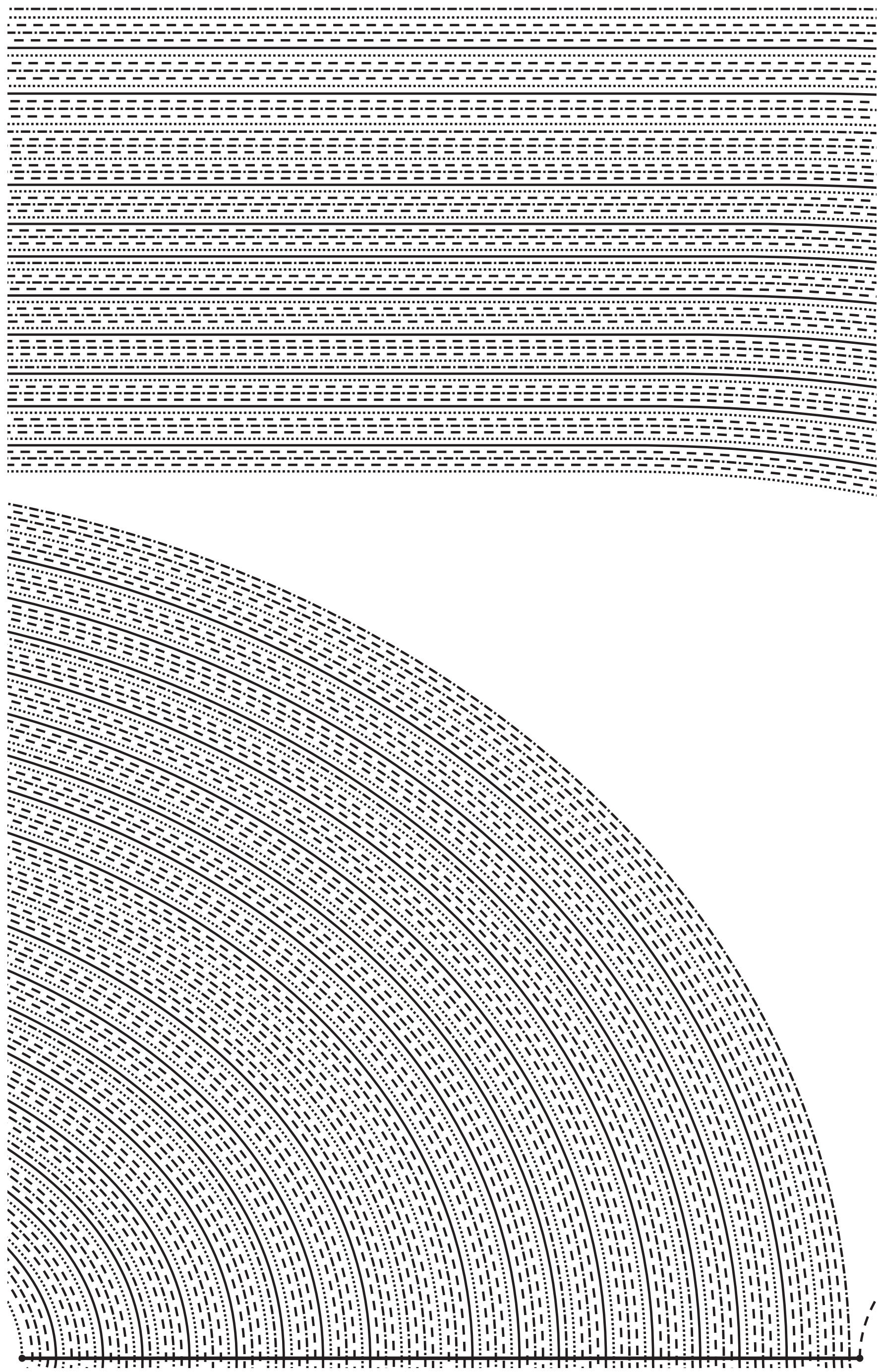
The part $(+, 2)$ of Figure 11.



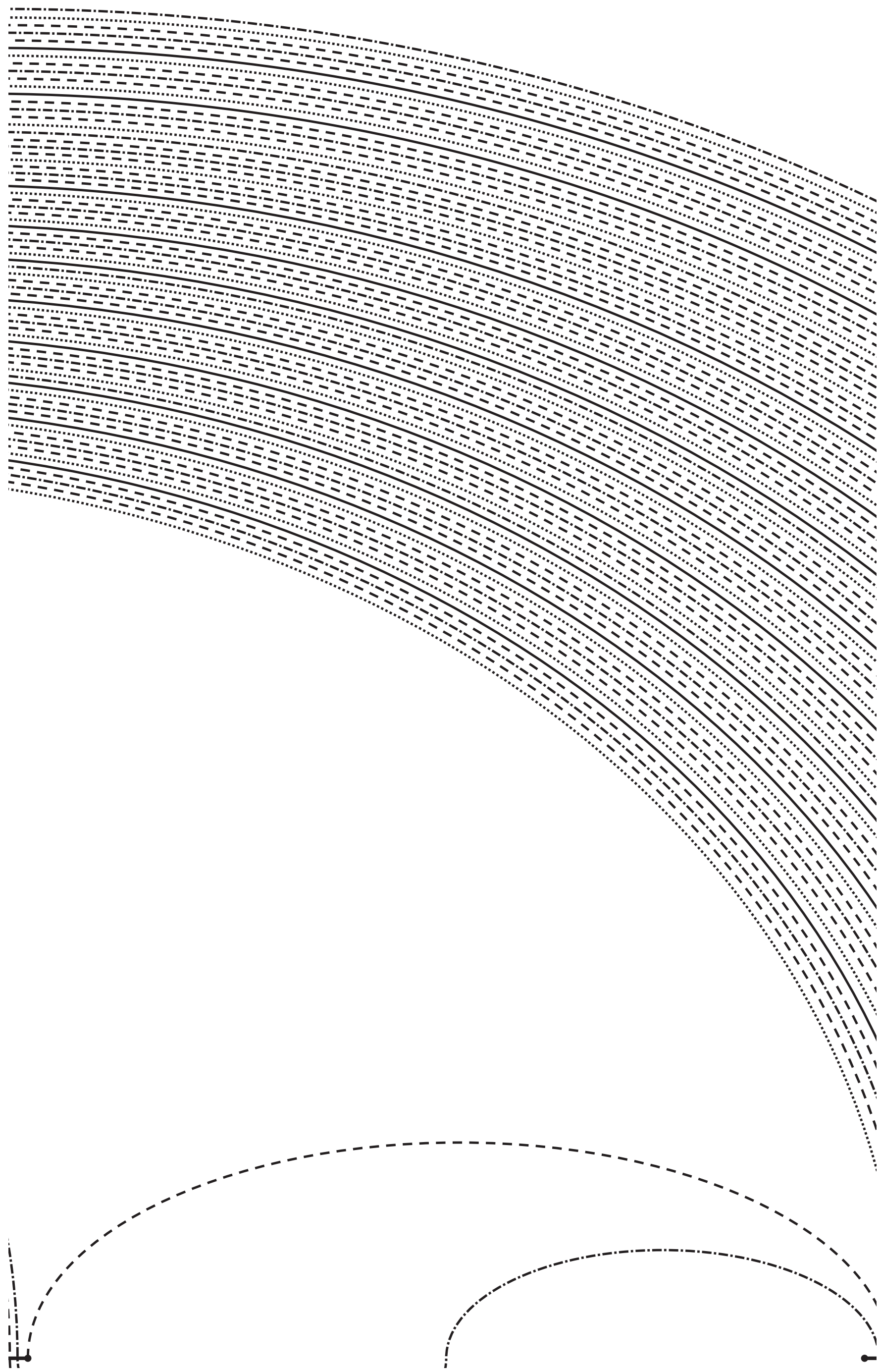
The part $(+, 3)$ of Figure 11.



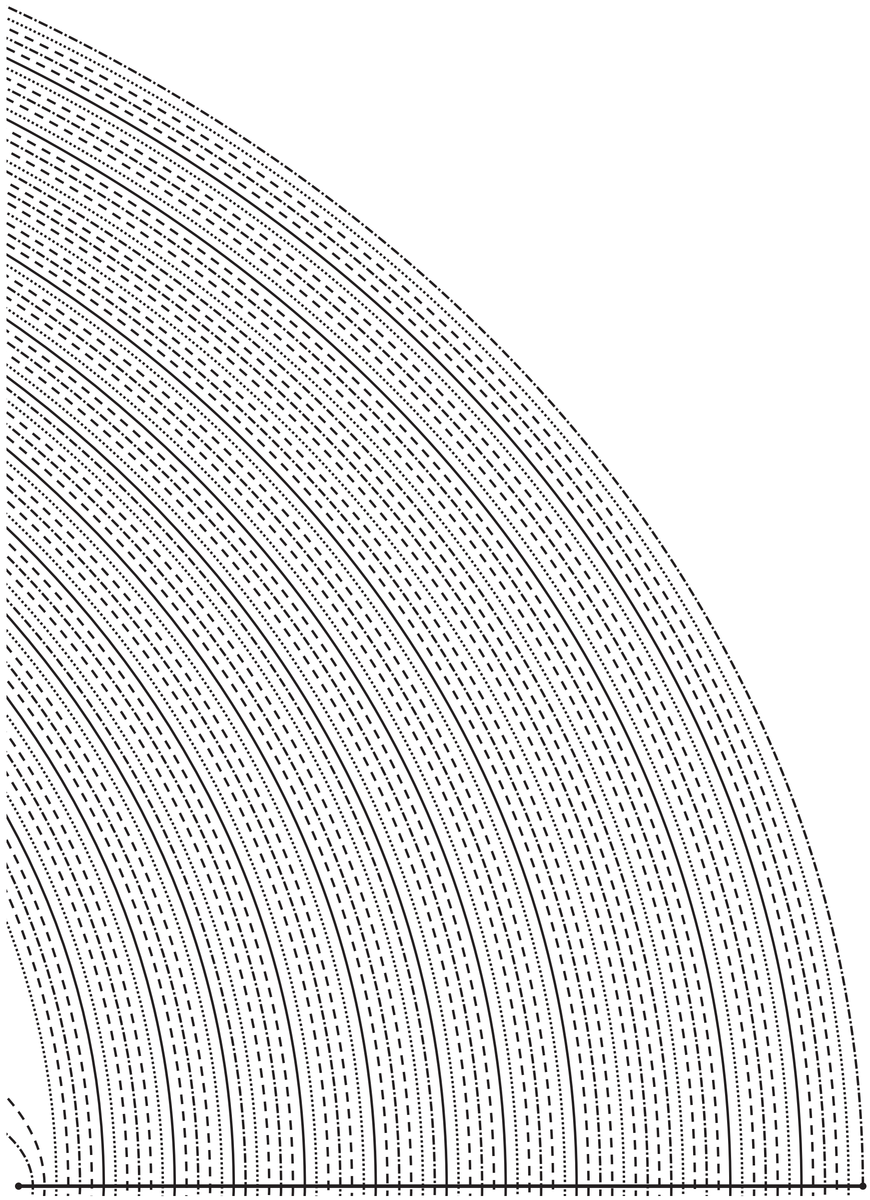
The part $(+,4)$ of Figure 11.



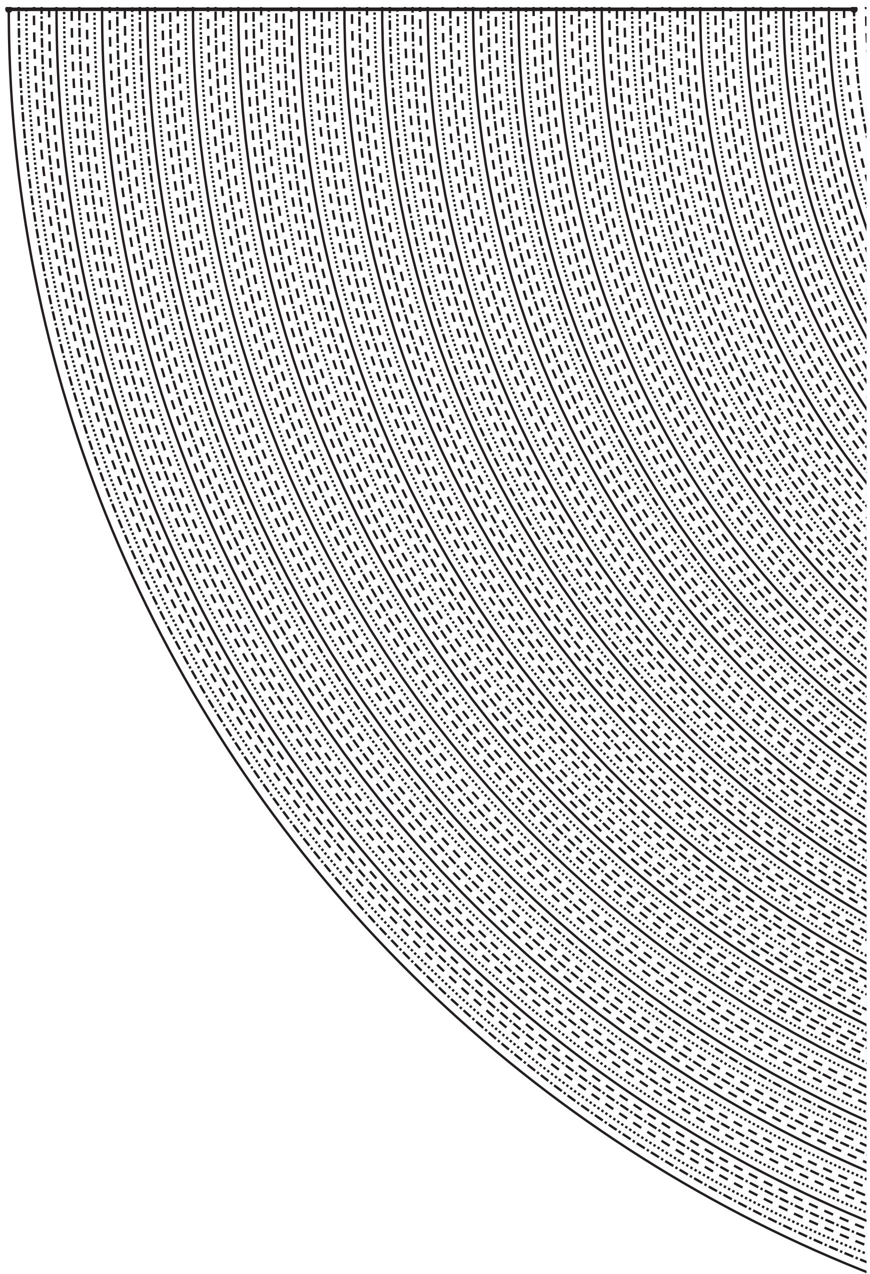
The part $(+, 5)$ of Figure 11.



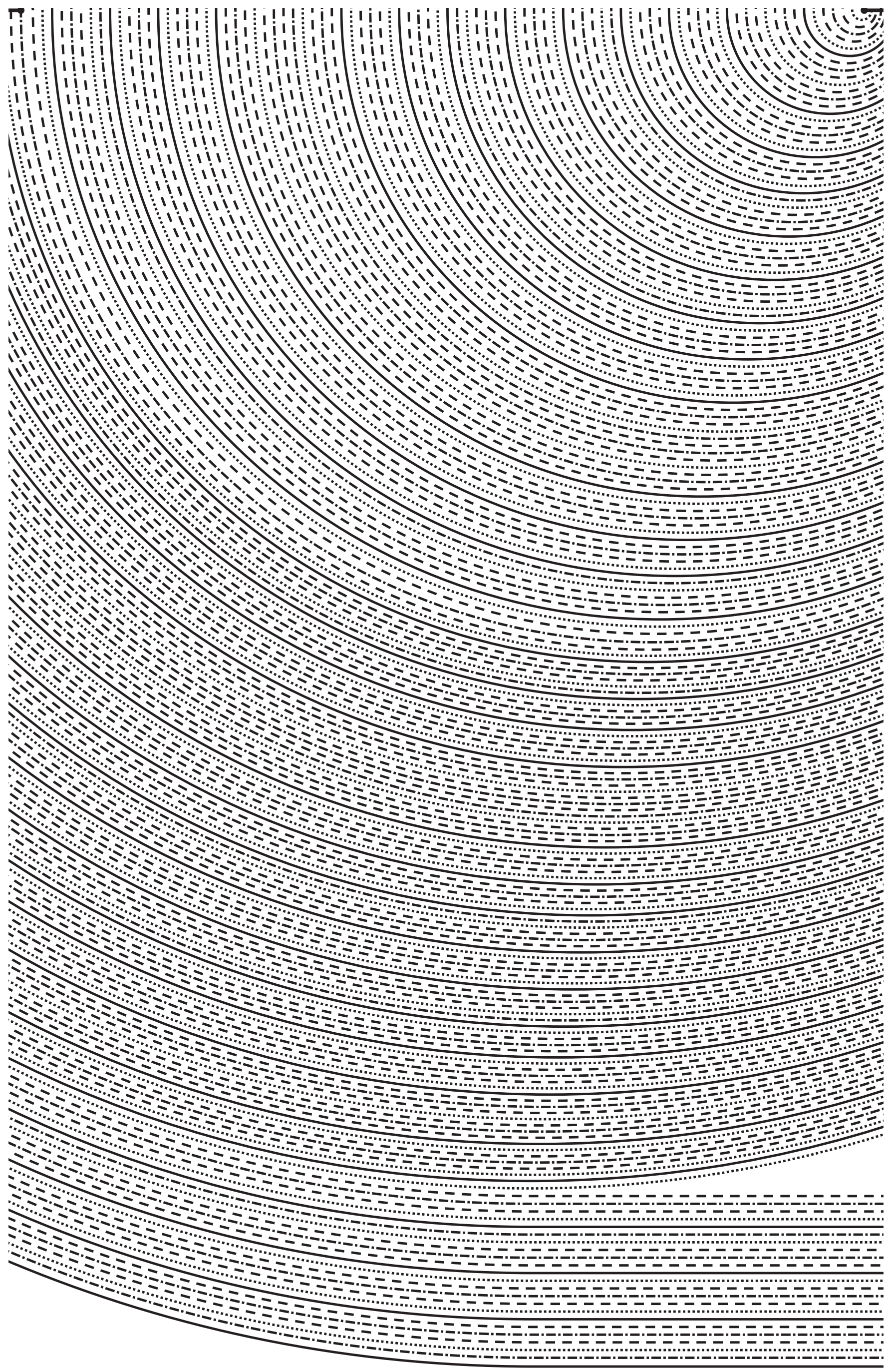
The part $(+, 6)$ of Figure 11.



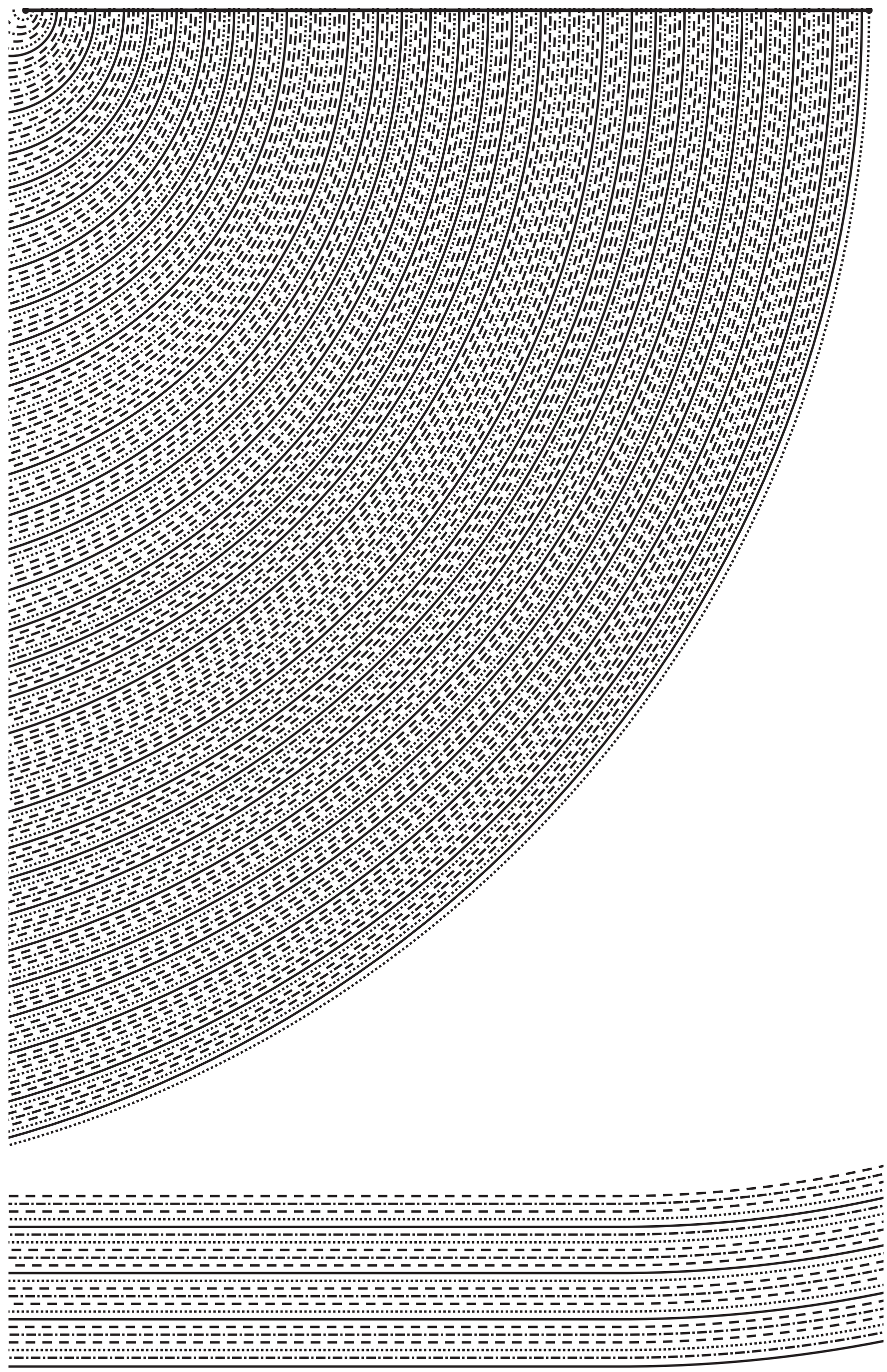
The part $(+, 7)$ of Figure 11.



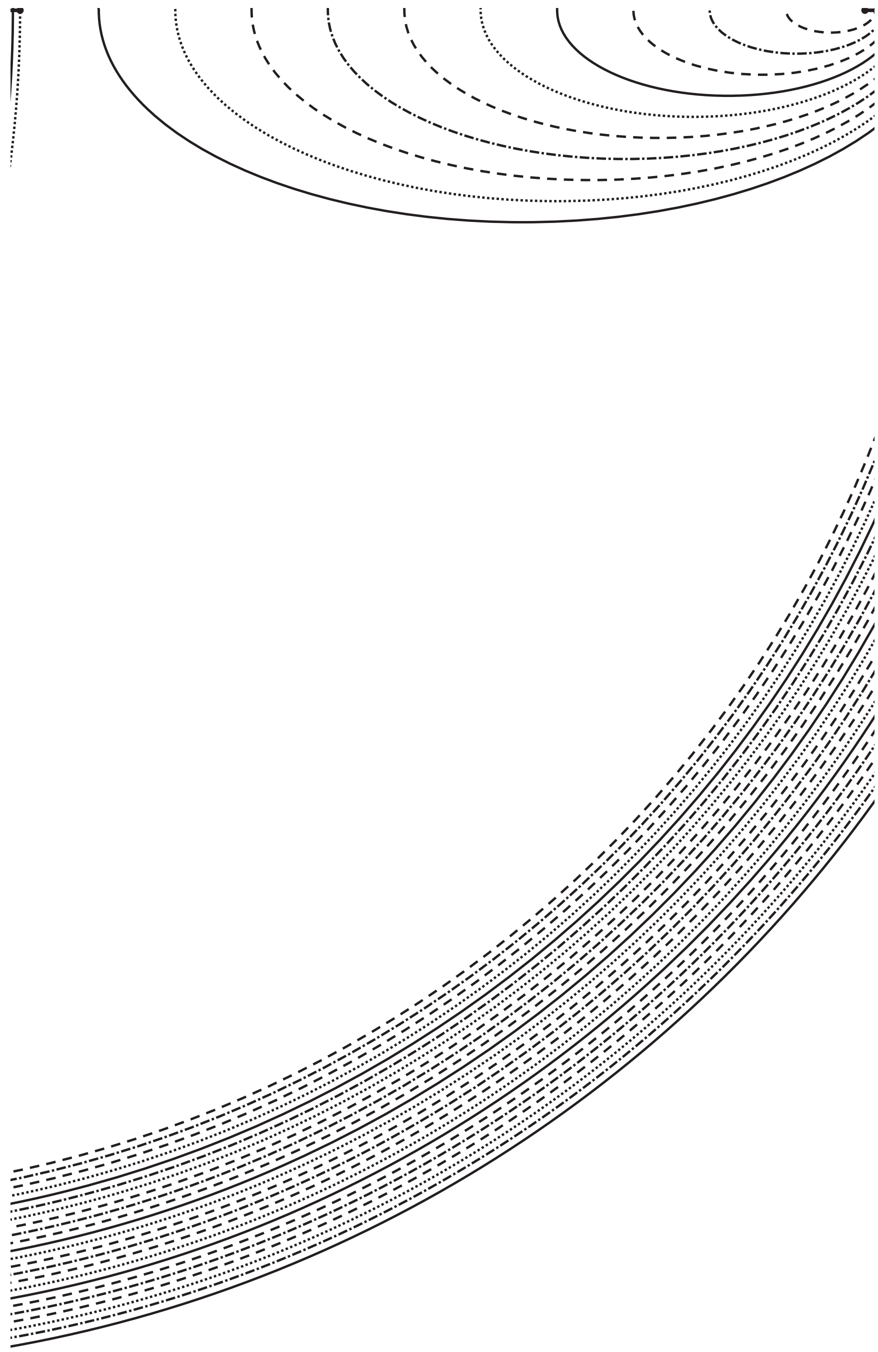
The part $(-,1)$ of Figure 11.



The part $(-, 2)$ of Figure 11.



The part $(-,3)$ of Figure 11.



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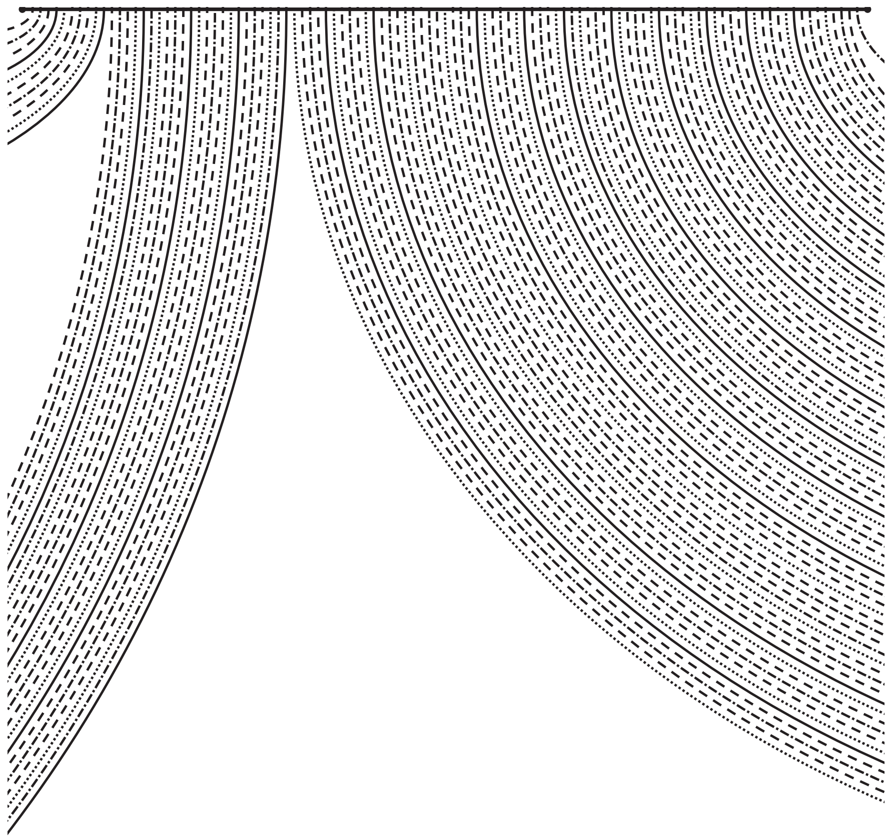

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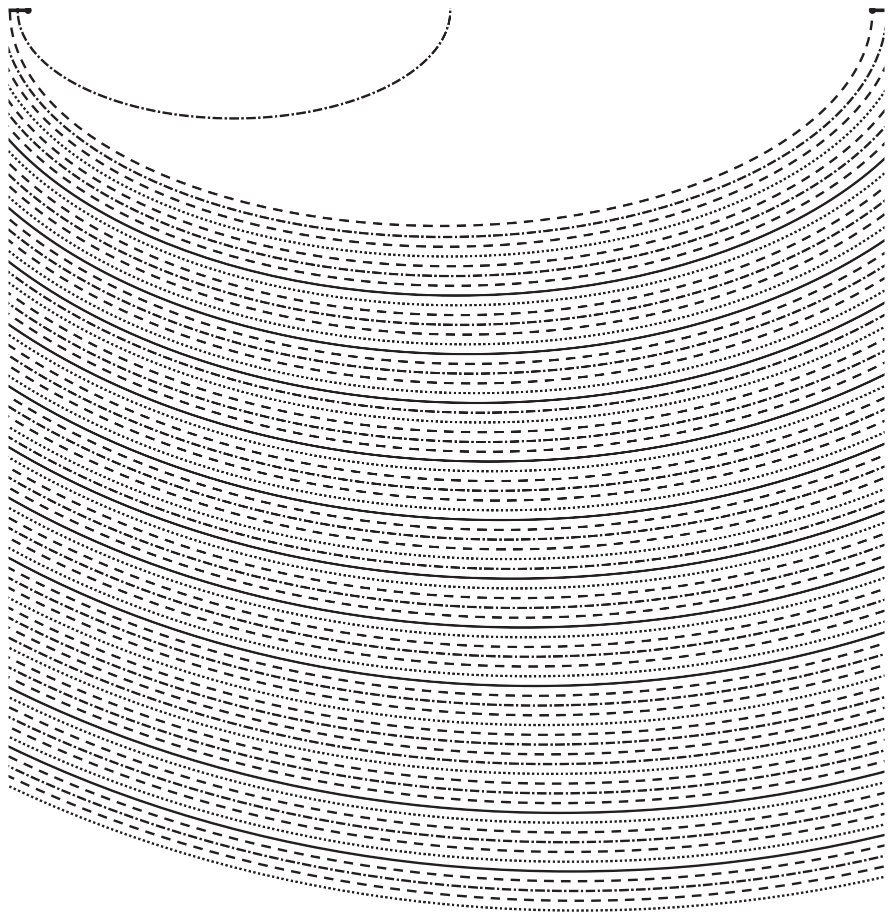
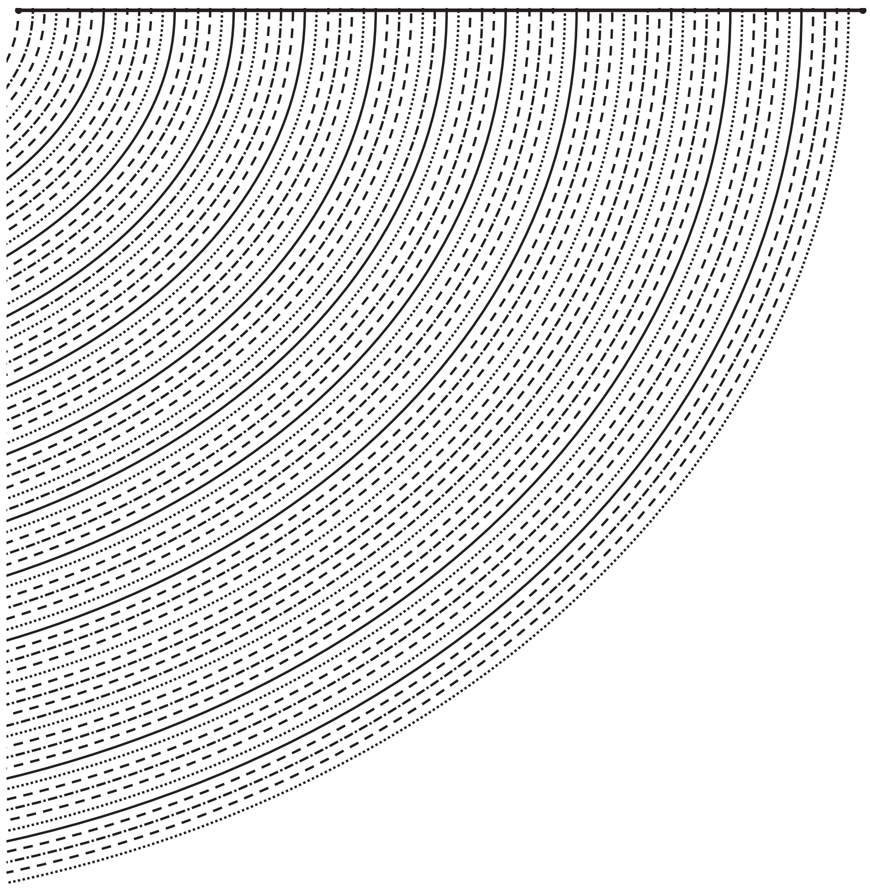


The part $(-, 6)$ of Figure 11.



The part $(-,7)$ of Figure 11.